

National Yang Ming Chiao Tung University
Department of Applied Mathematics
Probability Qualifying Examination

Date: September 8, 2026

Time: 8:30-12:30

Instructions

This is a closed-book examination. There are 10 problems, and you should **submit solutions to exactly 6 problems**. On the first page of your exam paper, clearly indicate which 6 problems you have attempted.

Each problem you attempt is worth 20 points, for a maximum total of 120 points. **A total score of 80 points** is required to pass.

If you believe that a problem is misstated or ambiguous, interpret it in a reasonable way that makes the problem nontrivial, and clearly state your interpretation.

Unclear or poorly explained solutions may receive no credit. It is your responsibility to present your answers clearly, rigorously, and in a proper mathematical form.

Problem (1) Suppose that X and Y are independent Gaussian random variables with mean zero and variance one. Let $0 < \alpha < 1$, define $Z = \alpha X + (1 - \alpha)Y$.

(a) What is the joint probability density function $f(x, z)$ of (X, Z) .

(b) Write $E[X|Z = z]$ in terms of $f(x, z)$ and use it to find the value of $E[X|Z = 0]$.

You may need the following result: If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then } A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Problem (2) Let X be an integrable random variable and $f(\xi) = E[e^{i\xi X}]$. Show that $f'(0) = iE[X]$.

Problem (3) Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with the common distribution

$$P\{X_1 = 1\} = P\{X_1 = -1\} = \frac{1}{2}.$$

Define

$$M_n = \sum_{k=1}^n \frac{1}{k} X_k.$$

(a) Show that M_n is a martingale with respect to $\mathcal{F}_n := \sigma\{X_1, \dots, X_n\}$.

(b) Show that M_n converges a.s. to a limit.

Problem (4) Suppose that $\mathcal{G}$ and $\mathcal{F}$ are σ -fields with $\mathcal{G} \subseteq \mathcal{F}$ and Y is a square-integrable random variable. Show that

$$E[|Y - E[Y|\mathcal{G}]|^2] \geq E[|Y - E[Y|\mathcal{F}]|^2].$$

Problem (5) Prove that given any sequence of random variables $\{X_n, n \geq 1\}$ whose range is $(-\infty, \infty)$, there exist constants $c_n \rightarrow \infty$ such that

$$P\left\{\lim_{n \rightarrow \infty} \frac{X_n}{c_n} = 0\right\} = 1.$$

Given a careful description of how you choose c_n .

Problem (6) Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with

$$P\{X_1 = -1\} = 1/3 \text{ and } P\{X_1 = 1\} = 2/3.$$

Define $S_0 = 0$ and

$$S_n = \sum_{k=1}^n X_k, \text{ for } n \geq 1.$$

Let

$$T := \inf\{n \geq 1 : S_n = -a \text{ or } S_n = b\},$$

where $a > 0$ and $b > 0$ are nature numbers. Show that $P\{T < \infty\} = 1$.

Problem (7) Suppose $\{X_n, n \geq 1\}$ are independent exponentially random variables with parameters $\lambda_n > 0$, i.e., $P\{X_n > x\} = e^{-\lambda_n x}$, $x \geq 0$. Prove that

$$\sum_{n=1}^{\infty} X_n \text{ converges a.s. if and only if } \sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty.$$

Problem (8) Let $\{X_n, n \geq 1\}$ be independent random variables such that

$$X_n = \begin{cases} \pm n, & \text{each with probability } (4n^2)^{-1}, \\ \pm 1, & \text{each with probability } 2^{-1} - (4n^2)^{-1}. \end{cases}$$

Define $S_n = \sum_{k=1}^n X_k$. Prove that

$$\frac{S_n}{\sqrt{n}} \text{ converges weakly to } N(0, \sigma^2) \text{ as } n \rightarrow \infty,$$

and compute σ .

Problem (9) Suppose that X is integrable and $\{\mathcal{F}_n, n \geq 1\}$ is a filtration. Prove that $\{E[X|\mathcal{F}_n], n \geq 1\}$ is uniformly integrable.

Problem (10)

A well-known result states that if a Markov chain is irreducible and recurrent, then it has a unique stationary distribution. Show that neither irreducibility nor recurrence can be omitted from this result by providing appropriate counterexamples.