

# NATIONAL YANG MING CHIAO TUNG UNIVERSITY

## 2026 ODE Ph.D. Qualifying Exam

Academic Year 115-1, 2026

1. (20 %) Consider the scalar equation

$$x'(t) = t x(t) + |x(t)|^\alpha, \quad x(0) = 0, \quad (1)$$

where  $\alpha > 0$ .

1. (10 pts.) Determine all  $\alpha > 0$  for which the standard Picard–Lindelöf theorem implies local uniqueness.
2. (10 pts.) For  $0 < \alpha < 1$ , use the substitution

$$u(t) = x(t)^{1-\alpha} \quad (2)$$

on intervals where  $x(t) > 0$ , and derive a linear differential equation for  $u$ . Then find an explicit formula for the nontrivial solutions that start moving at a prescribed waiting time  $\tau \geq 0$ .

2. (20 %) Let

$$f : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$$

be continuous. Assume that there exists a constant  $L > 0$  such that

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|$$

for every  $t \in [0, T]$  and  $x, y \in \mathbb{R}^n$ .

Suppose  $x(t)$  and  $y(t)$  are two solutions of

$$z'(t) = f(t, z(t))$$

with initial conditions

$$x(0) = x_0, \quad y(0) = y_0.$$

1. (8 pts.) Show that

$$\|x(t) - y(t)\| \leq \|x_0 - y_0\| + L \int_0^t \|x(s) - y(s)\| ds.$$

2. (8 pts.) Use Grönwall's inequality to prove that

$$\|x(t) - y(t)\| \leq e^{Lt} \|x_0 - y_0\|$$

for every  $t \in [0, T]$ .

3. (4 pts.) Deduce that if

$$x_0 = y_0,$$

then

$$x(t) = y(t)$$

for every  $t \in [0, T]$ . Hence prove uniqueness of the initial value problem

$$z'(t) = f(t, z(t)), \quad z(0) = z_0.$$

3. (20 %)  $x^* = (0, 0, 0)$  is a fixed point of the following system of differential equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2x_2 + x_2x_3 - x_1^3 \\ x_1 - x_1x_3 - x_2^3 \\ x_1x_2 - x_3^3 \end{bmatrix}.$$

1. (7 pts.) Show that  $x^* = (0, 0, 0)$  is a nonhyperbolic fixed point.

2. (7 pts.) Is the function

$$V(x_1, x_2, x_3) := x_1^2 + 2x_2^2 + x_3^2$$

a (strict) Lyapunov function at the origin  $x^* = (0, 0, 0)$  for this system of differential equations?

3. (6 pts.) Is the fixed point  $x^* = (0, 0, 0)$  a sink (i.e., asymptotically Lyapunov stable)?

4. (20 %) Consider the system of differential equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 - 9x_1(x_1^2 + x_2^2) \\ x_1 - x_2 - 9x_2(x_1^2 + x_2^2) \end{bmatrix}.$$

Does this system have a periodic orbit? Note: a fixed point is not considered as being a periodic solution.

5. (20 %) Consider the system

$$\begin{aligned} \dot{x} &= x(1 - r) - y, \\ \dot{y} &= y(1 - r) + x, \end{aligned}$$

where  $r^2 = x^2 + y^2$ . This system has a periodic solution

$$\gamma(t) = (\cos(t), \sin(t))^T.$$

Compute the Floquet multipliers of  $\gamma(t)$  to determine its stability.

*Hint: Polar coordinate transformation may help.*