

NYCU Department of Applied Mathematics
Qualifying Examination in Algebra
for the Ph.D. Program

September, 2026

Show all work and justify every step. If you invoke a theorem, state precisely which result you are using (name and source if relevant). Answers without explanation will receive no credit.

1. (a) (15 pts) Prove that every group of order 45 is abelian.
(b) (10 pts) Classify all abelian groups (up to isomorphism) of order 1800.
2. (a) (15 pts) Let p be a prime number, and let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x], \quad n \geq 1.$$

Suppose that $a_n \not\equiv 0 \pmod{p}$, but $a_i \equiv 0 \pmod{p}$ for all $0 \leq i < n$, with $a_0 \not\equiv 0 \pmod{p^2}$. Show that $f(x)$ is irreducible over $\mathbb{Q}$.

- (b) (10 pts) Show that $x^4 - x^3 + x^2 - x + 1$ is irreducible over $\mathbb{Q}$.
3. Let $E = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ be a finite field extension over $\mathbb{Q}$, and let $\alpha = \sqrt{6} + \sqrt{10} + \sqrt{15}$.
 - (a) (5 pts) Show that $E/\mathbb{Q}$ is a Galois extension.
 - (b) (5 pts) Find the group structure of the Galois group $\text{Gal}(E/\mathbb{Q})$.
 - (c) (5 pts) Find the minimal polynomial of α over $\mathbb{Q}$ by considering the action of $\text{Gal}(E/\mathbb{Q})$ on α .
 - (d) (5 pts) Show that $\sqrt{6}, \sqrt{10}, \sqrt{15} \in \mathbb{Q}(\alpha)$ by considering α, α^2 and α^3 .
 - (e) (5 pts) Find the minimal polynomial of $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}(\alpha)$.
4. Let F be a finite field with characteristic p .
 - (a) (5 pts) Show that the map $\sigma_p: F \rightarrow F$ defined by $\sigma_p(a) = a^p$ for $a \in F$ is a field automorphism.
 - (b) (5 pts) Show that the fixed field of σ_p is the finite field $\mathbb{Z}_p$.
 - (c) (10 pts) Show that the multiplicative group $(F^*, \cdot)$ of nonzero elements of F is cyclic.
 - (d) (5 pts) Show that a finite extension E of F is a simple extension of F .