PhD Qualifying Exam in Numerical Analysis

Fall, 2025

Please do any **FIVE** of the following problems only. Each problem counts 20 points.

1. Let $A \in \mathbb{R}^{n\times n}$ be a strictly diagonally dominant matrix by rows and $\mathbf{b} \in \mathbb{R}^n$. Prove that the Gauss-Seidel method for the solution of the linear system $A\mathbf{x} = \mathbf{b}$ is convergent.

2. Given $n+1$ distinct points $x_0, \dots, x_n$ with $n \in \mathbb{N}$. Let $\ell_i \in \mathbb{P}_n$ with $\ell_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$ and $\ell_i(x_j) = \delta_{ij}$, $i = 0, \dots, n$.

   Prove that the polynomials $\{\, \ell_i,\ i = 0, \dots, n \,\}$ form a basis for $\mathbb{P}_n$.

3. Let $f \in C^2([a,b])$.
   (a) Find the quadrature error of the midpoint formula, $E_0(f)$.
   (b) Find the quadrature error of the trapezoidal formula, $E_1(f)$.
   (c) Prove that $|E_1(f)| \simeq 2|E_0(f)|$.

4. Fix $0 < T < +\infty$. Given an initial-value problem (IVP): $y'(t) = f(t, y(t))$, $t \in I$ and $y(t_0) = y_0$, where $f(t,y)$ is a continuous real-valued function in $I \times (-\infty, +\infty)$ and $I = (t_0, t_0 + T)$. For $h > 0$, let $t_n = t_0 + nh$, with $n = 0, 1, 2, \dots, N_h$, be the sequence of discretization nodes of I into subintervals $I_n = [t_n, t_{n+1}]$, where N_h is maximum integer such that $t_{N_h} \leq t_0 + T$. Prove that Heun's method has order 2 with respect to h.

5. Given a two-point boundary value problem (BVP): $-u''(x) = f(x)$, $0 < x < 1$ and $u(0) = u(1) = 0$, where $f \in C^0([0,1])$. Consider that the approximation to the solution u of the BVP is a finite sequence $\{u_j\}_{j=0}^n$ defined only at grid points $\{x_j\}_{j=0}^n$ with $x_j = jh$ and $h = 1/n$ for $n \geq 4$. Using the centred finite difference for $u''(x_j)$, $j = 1, \dots, n-1$, obtains the finite difference approximation of the BVP, $\mathrm{A}_{\mathrm{fd}}\mathbf{u} = \mathbf{f}$, where A_{fd} is the symmetric $(n-1) \times (n-1)$ finite difference matrix, $\mathbf{u} = (u_1, \dots, u_{n-1})^\top$, and $\mathbf{f} = (f_1, \dots, f_{n-1})^\top$ with $f_j = f(x_j)$. Prove that the matrix A_{fd} is an M-matrix.

6. Let I be an interval of $\mathbb{R}$ containing the point t_0. Consider the scalar Cauchy problem, that is, to find a real-valued function $y \in C^1(I)$ such that

   $$\begin{cases} y'(t) = f(t, y(t)), & t \in I, \\ y(t_0) = y_0, \end{cases}$$

   where $f(t,y)$ is a given real-valued function in $I \times (-\infty, +\infty)$. Apply the θ-method to the approximate solution of the scalar Cauchy problem.
   (a) Analyze the local truncation error for $\theta \neq \frac{1}{2}$.
   (b) Analyze the local truncation error for $\theta = \frac{1}{2}$.