PhD Qualifying Exam in Methods of Applied Mathematics

Fall 2025

Show all work clearly and no credits will be given for only answers.

Problem 1 (18 points) Consider the following initial boundary value problem

$$u_t = Du_{xx} + ru(1 - \frac{u}{K})(\frac{u}{B} - 1), \quad 0 < x < H, t > 0, \tag{1}$$

$$u_x(0,t) = u_x(H,t) = 0, \quad t > 0, \tag{2}$$

$$u(x,0) = A\cos(\frac{2\pi x}{H}), \quad 0 < x < H, \tag{3}$$

where $u(x,t)$ represents the population of some substance, x is in length, t is in time, the parameters D, r, K, B are positive constants. Please list the dimensions of all nine variables and parameters and nondimensionalize the equation.

Problem 2 (18 points) Consider the boundary value problem

$$\epsilon y'' + xy' - xy = 0, \quad 0 < x < 1, \tag{4}$$

$$y(0) = 0, y(1) = e, \tag{5}$$

where $0 < \epsilon << 1$ is a small parameter. Find the leading order uniform approximation of the solution and sketch your solution. Please label your plot properly.

Problem 3 (18 points) Determine the natural boundary condition at $x = b$ for the variational problem $J(y)$ defined by

$$J(y) := \int_a^b L(x, y, y')\, dx + G(y(b)),$$

where $y \in C^2[a,b]$ is a function of x satisfying $y(a) = y_0$, the function L is assumed to be twice continuously differentiable in each of its three arguments and G is a given differentiable function on $\mathbb{R}$.

Problem 4 (18 points) Determine if there is a Green's function associated with the operator $Lu = u'' + 4u, 0 < x < \pi$ with $u'(0) = u'(\pi) = 0$. Find the solution to the boundary problem

$$u'' + 4u = f(x), 0 < x < \pi, u'(0) = u'(\pi) = 0,$$

where $f(x)$ is assumed to be smooth.

Problem 5 (10 points) Use the energy method to show that the solution to the following initial boundary value problem

$$u_t - ku_{xx} = 0, \quad 0 < x < l, 0 < t < T, \tag{6}$$

$$u(x,0) = u_0(x), \quad 0 < x < 1, \tag{7}$$

$$u_x(0,t) = 0, u(l,t) = h(t), \quad 0 < t < T, \tag{8}$$

must be unique where k, l and T are positive constants.

Problem 6 (18 points) Consider the differential-integral operator

$$Ku := -u'' + 4\pi^2 \int_0^1 u(s)\, ds, \quad u'(0) = u(1) = 0.$$

Show that the eigenvalues of the operator K, provided they exist are positive. Find the eigenfunctions corresponding to the eigenvalue $\lambda = 4\pi^2$.