# NYCU DAM Qualifying Examination in Discrete Mathematics for the Ph.D. Program

Sep, 2025

**Note:** Show your work and justify your answers as clear as possible for full credits.

1. (10 %) Show directly that the complete bipartite graph $K_{3,3}$ is not planar without applying Kuratowski's theorem or Euler's formula.

2. The chromatic polynomial $\chi_G(x)$ of a finite (not necessarily simple) graph $G$ is the number of proper colorings of $G$ with $x$ colors. For an edge $e$ in $G$, denote $G-e$ and $G/e$ the graphs obtained from $G$ by deleting $e$ and contracting $e$, respectively.
   (a) (5 %) Prove that for graph $G$ with at least one (possibly loop or parallel) edge, $\chi_G(x) = \chi_{G-e}(x) - \chi_{G/e}(x)$.
   (b) (10 %) Prove that the number of acyclic orientations, in which there are no directed cycles, of a graph $G$ on $n$ vertices is $(-1)^n \chi_G(-1)$.

3. (15 %) Prove that if the edges of $K_n$ are colored red or blue, then the number of monochromatic triangle is at least $\binom{n}{3} - \left\lfloor \frac{n}{2} \left\lfloor (\frac{n-1}{2})^2 \right\rfloor \right\rfloor$.

4. A network is a finite directed graph $D = (V, A)$ together with two distinguished vertices the source $s$ and the terminal $t$, and a non-negative real-valued function the capacity $c$ on $A$. A cut $(X, Y)$ is a partition of $V$ such that $s \in X$ and $t \in Y$.
   (a) (5 %) Describe the maxflow-mincut (Ford and Fulkerson) theorem.
   (b) (10 %) Let $(X_1, Y_1)$ and $(X_2, Y_2)$ be minimum cuts in a network. Show that $(X_1 \cup X_2, Y_1 \cap Y_2)$ is also a minimum cut.

5. (15 %) Prove that there are $\left\{\frac{n^2}{12}\right\}$, the number nearest to $\frac{n^2}{12}$, mutually incongruent triangles formed from three vertices of a regular $n$-gon.

6. Define $\mu(d) = (-1)^k$ if $d$ is the product of $k$ distinct primes, and 0 otherwise.
   (a) (5 %) Show that $\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$
   (b) (10 %) Determine the value of $\sum_{n \le x} \mu(n) \left\lfloor \frac{x}{n} \right\rfloor$ for real number $x$.

7. (a) (5 %) Show that the partitions of an $n$-set into pairs and singletons have a one-to-one correspondence to symmetric $n \times n$ permutation matrices.
   (b) (10 %) Find the exponential generating function for the number of symmetric $n \times n$ permutation matrices in a compact form.