# NYCU Department of Applied Mathematics
# Qualifying Examination in Algebra
# for the Ph.D. Program

September, 2025

**Show all work and justify every step. If you invoke a theorem, state precisely which result you are using (name and source if relevant). Answers without explanation will receive no credit.**

1. (a) (5 pts) Prove that no group of ordr $mp$ is simple where $p$ is a prime and $1 < m < p$.
   (b) (5 pts) Prove that no group of ordr 24 is simple.
   (c) (5 pts) Prove that no group of ordr 30 is simple.
   (d) (10 pts) Find all simple groups of order $\leqslant 30$.

2. Let $R = \mathbb{Z}[\sqrt{n}]$ where $n$ is a square-free positive integer.
   (a) (10 pts) Find a multiolicative norm $N$ on $R$ such that $|N(\alpha)| = 1$ for $\alpha \in R$ if and only if $\alpha$ is a unit of $R$.
   (b) (10 pts) Show that every $0 \neq \alpha \in R$ that is not a unit has a factorization into irreducibles in $R$.
   (c) (5 pts) Show that $\mathbb{Z}[\sqrt{2}]$ is a UFD.

3. For any prime $p \in \mathbb{N}$, consider the polynomial

$$\Phi_p(x) = \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \cdots + x + 1.$$

   Let $\zeta$ be a zero of $\Phi_p(x)$.
   (a) (5 pts) Determine $[\mathbb{Q}(\zeta) : \mathbb{Q}]$.
   (b) (5 pts) Find the group structure of the Galois group $G(\mathbb{Q}(\zeta)/\mathbb{Q})$.
   (c) (5 pts) Consider the splitting field $E$ of $x^p - 2 = 0$ over $\mathbb{Q}$. Show that $E = \mathbb{Q}(\sqrt[p]{2}, \zeta)$.
   (d) (5 pts) Determine $[\mathbb{Q}(\sqrt[p]{2}, \zeta) : \mathbb{Q}]$.
   (e) (5 pts) Show that $x^p - 2 = 0$ is irreducible over $\mathbb{Q}(\zeta)$.

4. Let $G$ be a finitely generated abelian group, and let $H$ be a subgroup of $G$.
   (a) (5 pts) Show that $H$ is finitely generated.
   (b) (5 pts) Show that $G$ is free of finite rank if and only if $G$ contains no nonzero elements of finite order.
   (c) (5 pts) Classify all $G$ (up to isomorphism) when $G$ has order 360.
   (d) (10 pts) Suppose that $G$ and $H$ are free, both of rank $n$. Fixing the isomorphisms $G \cong \mathbb{Z}^n$ and $H \cong \mathbb{Z}^n$, we have the following commutative diagram

$$\begin{array}{ccc} H & \hookrightarrow & G \\ \downarrow \wr & & \downarrow \wr \\ \mathbb{Z}^n & \xrightarrow{A} & \mathbb{Z}^n \end{array}$$

   where $A \in M_n(\mathbb{Z})$. Show that

$$[G : H] = |\det A|.$$